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Saint-Venant decay rates for a non-homogeneous isotropic mixture of elastic solids in anti-plane shear

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Abstract

The purpose of this research is to investigate the influence of material inhomogeneity on the decay of Saint-Venant end effects in anti-plane shear deformations of linear isotropic mixtures of elastic solids. The work is motivated by the recent research activity on functionally graded materials (FGMs), i.e. materials with spatially varying properties tailored to satisfy particular engineering applications. The spatial decay of solutions of a boundary value problem with variable coefficients on a semi-infinite strip is investigated. The results may be interpreted in terms of a Saint-Venant principle for anti-plane shear deformations of linear isotropic mixtures of elastic solids.

Keywords: Graded mixtures; Spatial decay rates; Inhomogeneous boundary problems

1. Introduction

The continuum theory of mixtures has been a subject of intensive study in recent years. The origin of the modern formulations of continuum thermomechanics theories of mixtures goes back to papers of Truesdell and Toupin (1960), Kelly (1964), Eringen and Ingram (1965), Ingram and Eringen (1967), Green and Naghdi (1965), Green and Naghdi (1968), Müller (1968) and Bowen and Wiese (1969).

In the theories for a mixture of elastic solids presented in Bowen (1976), Green and Steel (1966) and Steel (1967), the independent constitutive variables are the displacement gradients and the relative velocity, and the spatial description is used. The first theory for a mixture of elastic solids based on the Lagrangian description has been presented by Bedford and Stern (1972a,b). In this theory the independent constitutive

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variables are the displacement gradients and the relative displacement. In recent years an increasing interest has been developed in the study of the qualitative properties of this theory (Iesan and Quintanilla, 1994). It is worth noting that the model of interpenetrating solid continua was applied by Tiersten and Jahanmir (1977) to derive a theory of composites where the relative displacement of the individual constituents is infinitesimal.

There has been considerable recent interest investigating the influence of material inhomogeneity on the decay of Saint-Venant end effects in linear elasticity (Scalpato and Horgan, 1997; Chan and Horgan, 1998; Horgan and Payne, 1993; Horgan and Quintanilla, 2001a,b; Borrelli et al., 2004). The motivation for these studies has been provided by the recent research activity on functionally graded materials (FGMs), that is, materials with continuously varying properties tailored to satisfy specific engineering applications (see, e.g. the papers by Erdogan (1995), Pindera et al. (1997) and Aboudi et al. (1999) and the references cited therein).

The effects of material inhomogeneity on the decay of solutions to boundary value problems on a semi-infinite strip have been investigated by Scalpato and Horgan (1997), Chan and Horgan (1998), Horgan and Payne (1993) and Horgan and Quintanilla (2001a). The governing equations are second order elliptic partial differential equations with variable coefficients. This problem arises in the context of anti-plane shear deformations of inhomogeneous linearly elastic solid (see Horgan (1995) for a review of anti-plane shear). Scalpato and Horgan (1997), Chan and Horgan (1998), Horgan and Payne (1993) and Horgan and Quintanilla (2001a) showed that material inhomogeneity can have a significant influence on the decay of end effect. Horgan and Quintanilla (2001b) obtained analogous results for transient heat conduction.

Anti-plane shear deformations are one of the simpler deformations a material can undergo. In the papers of Scalpato and Horgan (1997), Chan and Horgan (1998) and Horgan and Quintanilla (2001a) shear for inhomogeneous linearly elastic materials was considered. In the paper of Scalpato and Horgan (1997) the attention was confined to laterally inhomogeneous materials, and so shear modulus was assumed to vary only with the transverse coordinate in the strip. The general inhomogeneous problem was considered by Chan and Horgan (1998). In the present paper we continue the work developed in these references. We consider the case that the material is a mixture of two elastic solids. The main propose of this paper is to study the rate of decay of anti-plane shear deformations of a mixture of two elastic solids. It is also worth nothing that when we study the decay of solutions in a mixture a natural question arises: What is the rate of decay of the solutions to the homogeneous distribution of displacement? This will be another particular aim of study in our paper.

As the motivation of this paper is the study of Saint-Venant end effects it is worth recalling the references of Horgan (1989, 1996) and Horgan and Knowles (1983). There, the history and current situation of this kind of studies is well described.

In Section 2 we state the boundary value problem that governs the anti-plane deformations for an isotropic, but non-homogeneous mixture of elastic solids. In particular, we set down the equations and boundary conditions for an isotropic mixture. In Section 3 we start the study of the rate of decay when the inhomogeneity depends on two spatial variables. We propose a change of variable which allows us to study certain families of boundary value problems representative of anti-plane deformations. Our change of variable is an extension of the change of variable proposed by Chan and Horgan (1998). This can not be done in the general case, but we can do it whenever the matrix which determines the inhomogeneity of the material (see (3.4)) satisfies a certain condition (see condition (3.5)). We also give some examples of matrices satisfying this condition. For instance, when all the coefficients of the matrix have the same kind of inhomogeneity or the matrix is diagonal the conditions hold. In Section 4 we consider the family when all the coefficients of the matrix have the same kind of inhomogeneity (see condition (3.6)). For this family we prove that the rate of decay is governed by the function $-\mu^{-1/2}\triangle\mu^{1/2}$($\mu$ is the function that defines the inhomogeneity) which is the same kind of behavior as that found in the usual elasticity. Another family (defined in (3.7)) is the object of study of Section 5. In this situation we have basically two inhomogeneities, one for each phase. First we prove that if both inhomogeneities have a positive (or negative) Laplacian, the behavior is similar to the usual elasticity. But when the product of the Laplacians is negative, then there is no determined behavior and further analysis is necessary. We also consider some families of examples and we give some estimates of the asymptotic behavior. Section 6 is devoted to the study of the decay to the homogeneous displacement for the boundary value problem proposed in Section 6. This problem proposes the study of a certain partial differential equation (see Eq. (6.3)). We distinguish three kind of examples which depend on the inhomogeneity of the functions. We also include a last section with the conclusions.

2. Anti-plane shear deformations

Let us consider a mixture of two constituent elastic solids s_1 and s_2. The mixture is viewed as a superposition of two continua, each following its own motion; each place in the mixture is occupied simultaneously from one particle of each constituent. It is worth noting that there are different proposition of mixtures of elastic materials. Here we restrict our attention to that proposed by Iesan (1997). In this case the free energy, the first Piola–Kirchhoff partial stress associated with each constituent and the internal body force is assumed to be functions of the gradients of the deformation (of both constituents) and the relative displacement. This theory is well stated by means of the usual axioms of the continuum mechanics from a general nonlinear point of view and in the case of heat conduction with two different temperatures. It is not necessary to recall here the whole presentation of this theory because the reader can find it in the work of Iesan (1997). On the other hand, we will restrict our attention to the linear theory and the study of the static problem. Nevertheless, we can recall that in this case the partial derivatives of the free energy with respect of the gradients of deformation and the relative displacement give the first Piola–Kirchhoff partial stress associated with each constituent and the internal body force. In the linear theory a constitutive equation for the free energy is assumed in terms of the displacements of each constituent and the relative displacement. We shall denote **u** and **w** the displacements of each constituent and we define the geometric tensors

$$e_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}), \quad f_{ij} = \frac{1}{2}(w_{i,j} + w_{j,i}), \quad d_i = u_i - w_i. \tag{2.1}$$

The constitutive equation for the free energy is (see Iesan (1997, p. 160))

$$F_{\mathrm{E}} = \frac{1}{2}(A_{ijrs}e_{ij}e_{rs} + C_{ijrs}f_{ij}f_{rs} + a_{ij}d_i d_j) + B_{ijrs}e_{ij}f_{rs} + D_{ijr}e_{ij}d_r + E_{ijr}f_{ij}d_r.$$

We will denote by **t** and **s** first Piola–Kirchhoff partial stress associated with each constituent and by **p** the internal body force. We know that (see Iesan (1997, p. 162))

$$t_{ij} = \frac{1}{2}\left(\frac{\partial F_{\mathrm{E}}}{\partial e_{ij}} + \frac{\partial F_{\mathrm{E}}}{\partial e_{ji}}\right); \quad s_{ij} = \frac{1}{2}\left(\frac{\partial F_{\mathrm{E}}}{\partial f_{ij}} + \frac{\partial F_{\mathrm{E}}}{\partial f_{ji}}\right); \quad p_i = \frac{\partial F_{\mathrm{E}}}{\partial d_i}.$$

If we assume that there is no supply terms the equilibrium equations of the static problem are

$$t_{ij,j} - p_i = 0, \quad s_{ij,j} + p_i = 0. \tag{2.2}$$

In view of the previous relations concerning the free energy and the first Piola–Kirchhoff partial stress associated with each constituent and the internal body force, the constitutive equations in the linear theory are

$$t_{ij} = A_{ijrs}e_{rs} + B_{ijrs}f_{rs} + D_{ijr}d_r, \tag{2.3}$$

$$s_{ij} = B_{rsij}e_{rs} + C_{ijrs}f_{rs} + E_{ijr}d_r, \tag{2.4}$$

$$p_i = D_{mni}e_{mn} + E_{mni}f_{mn} + a_{ij}d_j. \tag{2.5}$$

We assume that the material can be inhomogeneous. Thus the tensors $A_{ijrs}, B_{ijrs}, \ldots, a_{ij}$ depend on the point material $\mathbf{x}$. The following symmetries are satisfied (see Iesan, 1997, p. 161)

$$A_{ijrs} = A_{rsij} = A_{jirs}, \quad B_{ijrs} = B_{jirs} = B_{ijsr}, \tag{2.6}$$

$$C_{ijrs} = C_{rsij} = C_{jirs}, \quad D_{ijr} = D_{jir}, \quad E_{ijr} = E_{jir}, \quad a_{ij} = a_{ji}. \tag{2.7}$$

The constitutive equations take the following form:

$$t_{ij} = A_{ijrs}u_{r,s} + B_{ijrs}w_{r,s} + D_{ijr}(u_r - w_r), \tag{2.8}$$

$$s_{ij} = B_{rsij}u_{r,s} + C_{ijrs}w_{r,s} + E_{ijr}(u_r - w_r), \tag{2.9}$$

$$p_i = D_{mni}u_{m,n} + E_{mni}w_{m,n} + a_{ij}(u_j - w_j). \tag{2.10}$$

We consider Dirichlet boundary conditions

$$u_i = \tilde{u}_i, \quad w_i = \tilde{w}_i. \tag{2.11}$$

Here $\tilde{u}_i, \tilde{w}_i$ are given function on the boundary (see Fig. 1).

From now on, we assume that the constitutive tensors A_{ijrs}, $B_{ijrs}, \ldots, a_{ij}$ depend on the variables x_1 and x_2 but they are free of the variable x_3. If we assume that the solutions of the problem have the following form:

$$u_1 = u_2 = 0, \quad w_1 = w_2 = 0, \quad u_3 = u_3(x_1, x_2), \quad w_3 = w_3(x_1, x_2), \tag{2.12}$$

we obtain

$$t_{ij} = A_{ij3\alpha}u_{3,\alpha} + B_{ij3\alpha}w_{3,\alpha} + D_{ij3}(u_3 - w_3), \tag{2.13}$$

$$s_{ij} = B_{3\alpha ij}u_{3,\alpha} + C_{ij3\alpha}w_{3,\alpha} + E_{ij3}(u_3 - w_3), \tag{2.14}$$

$$p_i = D_{3\alpha i}u_{3,\alpha} + E_{3\alpha i}w_{3,\alpha} + a_{i3}(u_3 - w_3). \tag{2.15}$$

If we consider the system of equilibrium equations, we have

$$(A_{3\beta3\alpha}u_{3,\alpha} + B_{3\beta3\alpha}w_{3,\alpha} + D_{3\beta3}(u_3 - w_3))_{,\beta} - D_{3\alpha3}u_{3,\alpha} - E_{3\alpha3}w_{3,\alpha} - a_{33}(u_3 - w_3) = 0, \tag{2.16}$$

$$(A_{\gamma\beta3\alpha}u_{3,\alpha} + B_{\gamma\beta3\alpha}w_{3,\alpha} + D_{\gamma\beta3}(u_3 - w_3))_{,\beta} - D_{\gamma\alpha3}u_{3,\alpha} - E_{\gamma\alpha3}w_{3,\alpha} - a_{\gamma3}(u_3 - w_3) = 0, \tag{2.17}$$

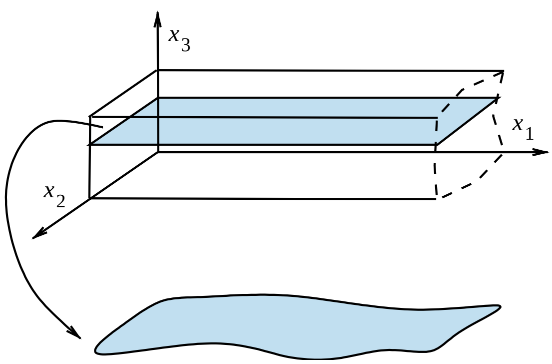

Fig. 1. Anti-plane deformation.

$$(B_{3\alpha 3\beta}u_{3,\alpha} + C_{3\beta 3\alpha}w_{3,\alpha} + E_{3\beta 3}(u_3 - w_3))_{,\beta} + D_{3\alpha 3}u_{3,\alpha} + E_{3\alpha 3}w_{3,\alpha} + a_{33}(u_3 - w_3) = 0, \tag{2.18}$$

$$(B_{3\alpha\gamma\beta}u_{3,\alpha} + C_{\gamma\beta 3\alpha}w_{3,\alpha} + E_{\gamma\beta 3}(u_3 - w_3))_{,\beta} + D_{\gamma\alpha 3}u_{3,\alpha} + E_{\gamma\alpha 3}w_{3,\alpha} + a_{\gamma 3}(u_3 - w_3) = 0. \tag{2.19}$$

Generally speaking, this system will be have no solutions because it has two variables and six equations. However if we restrict our considerations to materials such that four of the above equations are identically satisfied, we shall have a system with two variables and two equations.

From now on, we consider materials that satisfy

$$A_{\gamma\beta 3\alpha} = B_{\gamma\beta 3\alpha} = B_{3\alpha\gamma\beta} = C_{\gamma\beta 3\alpha} = D_{\gamma\beta 3} = E_{\gamma\beta 3} = a_{\gamma 3} = 0. \tag{2.20}$$

Materials which are isotropic and centrosymmetric do satisfy these conditions, so that our restrictions do not reduce the class of materials studied to the empty set. We introduce the notation

$$A_{\alpha\beta} = A_{3\alpha 3\beta}, \quad B_{\alpha\beta} = B_{3\alpha 3\beta}, \quad C_{\alpha\beta} = C_{3\alpha 3\beta}, \quad D_{\beta} = D_{3\beta 3}, \quad E_{\beta} = E_{3\beta 3}, \quad a = a_{33}, \tag{2.21}$$

$$u = u_3, \quad w = w_3. \tag{2.22}$$

So that we may write our system in the following form:

$$(A_{\beta\alpha}u_{,\alpha} + B_{\beta\alpha}w_{,\alpha} + D_{\beta}(u - w))_{,\beta} - D_{\alpha}u_{,\alpha} - E_{\alpha}w_{,\alpha} - a(u - w) = 0, \tag{2.23}$$

$$(B_{\alpha\beta}u_{,\alpha} + C_{\beta\alpha}w_{,\alpha} + E_{\beta}(u - w))_{,\beta} + D_{\alpha}u_{,\alpha} + E_{\alpha}w_{,\alpha} + a(u - w) = 0, \tag{2.24}$$

The symmetries (2.6) and (2.7) imply that

$$A_{\beta\alpha} = A_{\alpha\beta}, \quad C_{\beta\alpha} = C_{\alpha\beta}. \tag{2.25}$$

Our system can be written as

$$t_{\alpha,\alpha} - p = 0, \quad s_{\alpha,\alpha} + p = 0, \tag{2.26}$$

where

$$t_{\alpha} = A_{\alpha\beta}u_{,\beta} + B_{\alpha\beta}w_{,\beta} + D_{\alpha}(u - w), \tag{2.27}$$

$$s_{\alpha} = B_{\beta\alpha}u_{,\beta} + C_{\alpha\beta}w_{,\beta} + E_{\alpha}(u - w), \tag{2.28}$$

$$p = D_{\alpha}u_{,\beta} + E_{\alpha}w_{,\beta} + a(u - w). \tag{2.29}$$

The boundary conditions are:

$$u = \hat{u}, \quad w = \hat{w}. \tag{2.30}$$

From now on, we study the case of isotropic materials. Thus, we have

$$A_{11} = A_{22} = \mu_{11}, \quad B_{11} = B_{22} = \mu_{12}, \quad C_{11} = C_{22} = \mu_{22}, \tag{2.31}$$

$$A_{12} = B_{12} = B_{21} = C_{12} = D_1 = D_2 = E_1 = E_2 = 0. \tag{2.32}$$

The system of equations is

$$(\mu_{11}u_{,i} + \mu_{12}w_{,i})_{,i} - a(u - w) = 0, \tag{2.33}$$

$$(\mu_{12}u_{,i} + \mu_{22}w_{,i})_{,i} + a(u - w) = 0. \tag{2.34}$$

System (2.33) and (2.34) describes the classical anti-plane shear deformations in the case of an isotropic mixture of elastic solids.

3. A change of variable

In this section we study the rate of decay of the solutions of the problem determined by the system (2.33), (2.34) defined on the semi-infinite strip $(0,\infty)\times(0,1)$ when we impose the boundary conditions

$$u = w = 0,\ (x_1,x_2) \in [0,\infty) \times \{0,1\} \tag{3.1}$$

and the asymptotic conditions

$$u, w, u_{,i}, w_{,i} \to 0, \quad \text{as } x_1 \to \infty \quad \text{(uniformly)}. \tag{3.2}$$

It will be useful to introduce some notation. When we have a symmetric matrix

$$\mathbf{H} = \begin{pmatrix} a & b \\ b & c \end{pmatrix},$$

we denote by $M(\mathbf{H})$ the maximum eigenvalue and $m(\mathbf{H})$ the minimum eigenvalue. It is worth recalling that

$$M(\mathbf{H}) = \frac{(a+c) + \sqrt{(a-c)^2 + 4b^2}}{2}, \quad m(\mathbf{H}) = \frac{(a+c) - \sqrt{(a-c)^2 + 4b^2}}{2}. \tag{3.3}$$

In this section we assume that the matrix

$$\Lambda(x_1,x_2) = \begin{pmatrix} \mu_{11}(x_1,x_2) & \mu_{12}(x_1,x_2) \\ \mu_{12}(x_1,x_2) & \mu_{22}(x_1,x_2) \end{pmatrix} \tag{3.4}$$

satisfies the equality

$$\frac{\partial}{\partial x_i}\Lambda^{1/2} + \Lambda\frac{\partial}{\partial x_i}\Lambda^{-1/2} = \mathbf{0}. \tag{3.5}$$

Example 3.1. It is worth giving examples of matrices satisfying the condition (3.5). A family of them could be

$$\Lambda = \mu(x_1,x_2)\begin{pmatrix} k_{11} & k_{12} \\ k_{12} & k_{22} \end{pmatrix}, \tag{3.6}$$

where k_{ij} are constants. This case corresponds to an inhomogeneity which is similar in both constituents. A second family of examples is

$$\Lambda = \begin{pmatrix} \mu_{11}(x_1,x_2) & 0 \\ 0 & \mu_{22}(x_1,x_2) \end{pmatrix}. \tag{3.7}$$

In this second case the inhomogeneity is different for each phase. When the matrix Λ satisfies the condition (3.5), we can consider the change

$$\begin{pmatrix} u \\ w \end{pmatrix} = \Lambda^{-1/2}\begin{pmatrix} U \\ W \end{pmatrix}. \tag{3.8}$$

We have that

$$\begin{pmatrix} u_{,i} \\ w_{,i} \end{pmatrix} = (\Lambda^{-1/2})_{,i}\begin{pmatrix} U \\ W \end{pmatrix} + \Lambda^{-1/2}\begin{pmatrix} U_{,i} \\ W_{,i} \end{pmatrix}. \tag{3.9}$$

Thus

$$\Lambda\begin{pmatrix} u_{,i} \\ w_{,i} \end{pmatrix} = \Lambda(\Lambda^{-1/2})_{,i}\begin{pmatrix} U \\ W \end{pmatrix} + \Lambda^{1/2}\begin{pmatrix} U_{,i} \\ W_{,i} \end{pmatrix}. \tag{3.10}$$

Because of (3.5), it follows that:

$$\frac{\partial}{\partial x_i}\Lambda\begin{pmatrix} u_{,i} \\ w_{,i} \end{pmatrix} = [\Lambda(\Lambda^{-1/2})_{,i}]_{,i}\begin{pmatrix} U \\ W \end{pmatrix} + \Lambda^{1/2}\begin{pmatrix} U_{,ii} \\ W_{,ii} \end{pmatrix}. \tag{3.11}$$

The system (2.33), (2.34) can be transformed into

$$\begin{pmatrix} \Delta U \\ \Delta W \end{pmatrix} + \Lambda^{-1/2}[\Lambda(\Lambda^{-1/2})_{,i}]_{,i}\begin{pmatrix} U \\ W \end{pmatrix} + \Lambda^{-1/2}\begin{pmatrix} -a & a \\ a & -a \end{pmatrix}\Lambda^{-1/2}\begin{pmatrix} U \\ W \end{pmatrix} = \mathbf{0}. \tag{3.12}$$

Here the symbol Δ denotes the two dimensional Laplace operator.

If we denote by

$$\Gamma(x_1, x_2) = \Lambda^{-1/2}[\Lambda(\Lambda^{-1/2})_{,i}]_{,i} + \Lambda^{-1/2}\begin{pmatrix} -a & a \\ a & -a \end{pmatrix}\Lambda^{-1/2} \tag{3.13}$$

we can write the system as

$$\begin{pmatrix} \Delta U \\ \Delta W \end{pmatrix} + \Gamma(x_1, x_2)\begin{pmatrix} U \\ W \end{pmatrix} = \mathbf{0}. \tag{3.14}$$

The asymptotic behavior of the solutions (u, w) to the problem (2.33), (2.34) and (3.1) may now be deduced from the asymptotic behavior of solutions (U, W) to (3.14). This system is of the type of the Helmholtz equation with a variable matrix of coefficients. The exponential decay of the new system may be examined by using the energy method. Consider the case where the maximum eigenvalue of Γ can be positive. It can be shown that the "energy"

$$E(z) = \int_{R(z)} (U_{,i}U_{,i} + W_{,i}W_{,i} - (U, W)\Gamma(U, W))\mathrm{d}a, \tag{3.15}$$

where $R(z) = \{(x_1, x_2) \in (0, \infty) \times (0, 1), x_1 > z\}$, satisfies an inequality of the form

$$E(\zeta) \leqslant E(0)\exp(-2k\zeta), \tag{3.16}$$

where

$$k^2 = \pi^2\left(1 - \frac{\Gamma^+}{\pi^2}\right) \tag{3.17}$$

and

$$\Gamma^+ = \sup_{(x_1,x_2)\in[0,\infty)\times[0,1]} M(\Gamma(x_1, x_2)) < \pi^2. \tag{3.18}$$

A proof of the estimate (3.16) is based in the standard arguments on energy methods used to prove spatial decay. In this sense we define the function

$$P(z) = \int_{L(z)} (UU_{,1} + WW_{,1})\mathrm{d}x_2, \tag{3.19}$$

where $L(\zeta) = \{(x_1, x_2) \in (0, \infty) \times (0, 1), x_1 = \zeta\}$.

In view of the boundary conditions we have that

$$P(z+h) - P(z) = \int_z^{z+h} \int_{L(\zeta)} (U_{,i}U_{,i} + W_{,i}W_{,i} - (U,W)\Gamma(U,W))\,\mathrm{d}a \tag{3.20}$$

and

$$P'(z) = \int_{L(z)} (U_{,i}U_{,i} + W_{,i}W_{,i} - (U,W)\Gamma(U,W))\mathrm{d}x_2. \tag{3.21}$$

We note that

$$P'(z) \geqslant \int_{L(z)} \left(U_{,i}U_{,i} + W_{,i}W_{,i} - \Gamma^{+}(U^2+W^2)\right)\mathrm{d}x_2. \tag{3.22}$$

After the use of the arithmetic–geometric mean and the Poincaré inequalities, it follows that

$$\mid P(z) \mid \leqslant (2k)^{-1}P'(z). \tag{3.23}$$

Last inequality is well known in the study of the spatial decay estimates (see Flavin et al. (1989)). We conclude that either there exists z_0 such that $P(z_0) > 0$ and

$$P(z) \geqslant P(z_0)\exp(2k(z-z_0)), \tag{3.24}$$

for $z \geqslant z_0$, or we obtain that $P(z)$ tends to zero as z goes to infinite and we obtain the spatial decay estimate:

$$-P(z) \leqslant -P(0)\exp(-2kz). \tag{3.25}$$

This last inequality is the same of the estimate (3.16).

If we want that the estimate (3.16) holds it is sufficient that

$$\lim_{z\to\infty} \exp(-2kz)P(z) = 0. \tag{3.26}$$

We have proved the following result.

Theorem 3.1. *Let (u,w) be a solution of the problem determined by the system (2.33) and (2.34) the conditions (3.1) and the asymptotic condition (3.26), then the function $E(z)$ defined in (3.15) satisfies the estimate (3.16) where k is defined in (3.17).*

It could happen that our change of variable allows the existence of solutions satisfying condition (3.2) such that condition (3.26) does not hold. Then, as we want to study the solutions that satisfy condition (3.2), we need to control this aspect and to guarantee that we work with solutions which decay when z goes to infinity. But condition (3.26) is implied by condition (3.2) in a wide class of examples. Thus, some attention to this aspect would be necessary in our development. We will pay attention to this aspect in the examples.

When (3.16) holds, we can obtain estimates for the behavior of the solutions (u,w). We have that

$$F(\zeta) = \int_{L(\zeta)} (u^2+w^2)\mathrm{d}x_2 \leqslant K \sup_{L(\zeta)} M(\Lambda^{-1})\exp(-2k\zeta), \tag{3.27}$$

where k is given in (3.17) and K depends on the boundary conditions.

Before concluding this section, we think that it is worth noting what happen in case that the material is homogeneous. That is when the functions μ_{ij} and a are constants. We could apply the previous arguments to study this case, but we believe that it is better to remember the analysis proposed for the system (2.33) and (2.34) when the functions are constants in the work of Quintanilla (2001). That work analyzed, by means of the separation of variables, the problem of static distribution of heat for a mixture. The system of equations is the same as the one proposed here for the homogeneous case. The rate of decay agrees with the rate of decay for the Laplace equation. This is $k = \pi$.

4. Discussion: case (3.6)

In this section, we apply the arguments of the previous section to the case when the matrix Λ is defined by (3.6) with $\mu(x_1, x_2) > 0, x_1 \in [0, +\infty), x_2 \in [0, 1]$. In this case, we have

$$\Lambda^{1/2} = \mu^{1/2}(x_1, x_2)\begin{pmatrix} l_{11} & l_{12} \\ l_{12} & l_{22} \end{pmatrix}, \tag{4.1}$$

where the matrix $\mathbf{L} = (l_{ij})$ is a square root of the matrix of the k_{ij}. Indeed, there exist at least four square roots. Since the matrix of k_{ij} is symmetric and definite positive, there exists a base in which the matrix is diagonal with positive eigenvalues: d_{11} and d_{22}. Then we can consider, for example, one square root of the above matrix as the diagonal matrix with the positive elements $\sqrt{d_{11}}$ and $\sqrt{d_{22}}$. The matrix Γ defined in (3.13) is in this case:

$$\Gamma = -\mu^{-1/2}\Delta(\mu^{1/2})\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \mu^{-1}\mathbf{L}^{-1}\begin{pmatrix} -a & a \\ a & -a \end{pmatrix}\mathbf{L}^{-1}. \tag{4.2}$$

From the symmetry of k_{ij}, it follows that $\mathbf{L}$ and $\mathbf{L}^{-1}$ are symmetric. We denote

$$\mathbf{L}^{-1} = \begin{pmatrix} b & c \\ c & d \end{pmatrix}. \tag{4.3}$$

Thus

$$\mathbf{L}^{-1}\begin{pmatrix} -a & a \\ a & -a \end{pmatrix}\mathbf{L}^{-1} = a\begin{pmatrix} -(b-c)^2 & (c-b)(c-d) \\ (c-b)(c-d) & -(c-d)^2 \end{pmatrix} = a\begin{pmatrix} -A^2 & B \\ B & -C^2 \end{pmatrix}, \tag{4.4}$$

where $A = b - c, B = (c - b)(c - d)$ and $C = c - d$. We note that $A^2C^2 - B^2 = 0$. Finally,

$$\begin{aligned} \Gamma &= \begin{pmatrix} -\mu^{-1/2}\Delta\mu^{1/2} & 0 \\ 0 & -\mu^{-1/2}\Delta\mu^{1/2} \end{pmatrix} + \mu^{-1}a\begin{pmatrix} -A^2 & B \\ B & -C^2 \end{pmatrix} \\ &= \begin{pmatrix} -\mu^{-1/2}\Delta\mu^{1/2} - \mu^{-1}aA^2 & \mu^{-1}aB \\ \mu^{-1}aB & -\mu^{-1/2}\Delta\mu^{1/2} - \mu^{-1}aC^2 \end{pmatrix} \end{aligned} \tag{4.5}$$

The maximum eigenvalue of Γ is

$$\begin{aligned} M(\Gamma) &= \frac{1}{2}\left(-2\mu^{-1/2}\Delta\mu^{1/2} - \mu^{-1}a(A^2 + C^2) + \sqrt{[\mu^{-1}a(C^2 - A^2)]^2 + 4\mu^{-2}a^2B^2}\right) \\ &= \frac{1}{2}\left(-2\mu^{-1/2}\Delta\mu^{1/2} - \mu^{-1}a(A^2 + C^2) + \sqrt{(\mu^{-1}a)^2(C^2 + A^2)^2}\right) = -\mu^{-1/2}\Delta\mu^{1/2}. \end{aligned} \tag{4.6}$$

We emphasize the independence of $M(\Gamma)$ with respect to the function a.

Clearly, if $\Delta\mu^{1/2} \geqslant 0$, then $M(\Gamma) \leqslant 0$ and $\Gamma^+ \leqslant 0$. So the decay rate k of (3.17) is greater than or equal to that for Laplace's equation (or the homogeneous case). Otherwise, if $\Delta\mu^{1/2} \leqslant 0$, it follows $M(\Gamma) \geqslant 0$ and so $\Gamma^+ \geqslant 0$. Then the decay rate k is smaller than or equal to that for Laplace's equation (or the homogeneous case).

Table 1
Rate of decay for some functions

μ	k	
(1) $\mu_0(1+\alpha x_2)^{-1}$	$\left(\pi^2+\frac{3}{4}\alpha^2(1+\alpha)^{-2}\right)^{1/2}>\pi$	
(2) $\mu_0(1+\alpha x_2)$	$\left(\pi^2-\frac{1}{4}\alpha^2\right)^{1/2}<\pi$	
(3) $\mu_0(1+\alpha x_2)^2$	π	
(4) $\mu_0(1+\alpha x_2)^n$	$\left(\pi^2-\frac{1}{4}\alpha^2 n(2-n)\right)^{1/2}<\pi$,	if $n\in(0,2)$
	$\left(\pi^2+\frac{1}{4}\alpha^2 n(n-2)(1+\alpha)^{-2}\right)^{1/2}>\pi$,	if $n\notin[0,2]$
	π,	if $n\in\{0,2\}$
(5) $\mu_0\exp(\pm\alpha x_2)$	$\left(\pi^2+\frac{1}{4}\alpha^2\right)^{1/2}>\pi$	
(6) $\mu_0(1+\alpha x_1)^{-1}$	π	
(7) $\mu_0(1+\alpha x_1)$	$\left(\pi^2-\frac{1}{4}\alpha^2\right)^{1/2}<\pi$	
(8) $\mu_0(1+\alpha x_1)^2$	π	
(9) $\mu_0(1+\alpha x_1)^n$	$\left(\pi^2-\frac{1}{4}\alpha^2 n(2-n)\right)^{1/2}<\pi$,	if $n\in(0,2)$
	π,	if $n\notin(0,2)$
(10) $\mu_0\exp(\pm\alpha x_1)$	$\left(\pi^2+\frac{1}{4}\alpha^2\right)^{1/2}>\pi$	

In the particular case when $M(\Gamma)$ is a constant $\lambda\leqslant\pi^2$, then $\Gamma^+=\lambda$, and $\mu^{1/2}$ satisfies the Helmholtz equation $\Delta\mu^{1/2}+\lambda\mu^{1/2}=0$. Now (3.17) becomes $k^2=\pi^2-\lambda$. If we consider μ a function of one variable, for example $\mu=\mu(x_2), x_2\in[0,1]$, then our Helmholtz equation is reduced to an ordinary differential equation

$$(\mu^{1/2})''+\lambda\mu^{1/2}=0 \tag{4.7}$$

with solutions $\mu^{1/2}=\mu_0^{1/2}\exp(\pm\sqrt{-\lambda}x_2)$ when $\lambda\neq 0$, and $\mu^{1/2}=c_1x_2+c_2$ when $\lambda=0$. In the last case $\mu=\mu_0(1+\alpha x_2)^2$ and $k=\pi$. Here we recover the results of Chan and Horgan (1998).

In Tables 1 and 2 we compute the rate of decay k given by (3.17) for some examples of functions $\mu(x_1,x_2)>0, x_2\in[0,1]$, $x_1\in[0,+\infty)$.

Table 2
Rate of decay for some functions

μ	k	
(11) $\mu_0(1+\alpha x_1+\beta x_2)^{-1}$	π	
(12) $\mu_0(1+\alpha x_1+\beta x_2)$	$\left(\pi^2-\frac{1}{4}(\alpha^2+\beta^2)\right)^{1/2}<\pi$	
(13) $\mu_0(1+\alpha x_1+\beta x_2)^2$	π	
(14) $\mu_0(1+\alpha x_1+\beta x_2)^n$	$\left(\pi^2-\frac{1}{4}(\alpha^2+\beta^2)n(2-n)\right)^{1/2}<\pi$,	if $n\in(0,2)$
	π,	if $n\notin(0,2)$
(15) $\mu_0\exp(\pm\alpha x_1\pm\beta x_2)$	$\left(\pi^2+\frac{1}{4}\left(\alpha^2+\beta^2\right)\right)^{1/2}>\pi$	
(16) $\mu_0(1+\alpha x_1)\exp(\pm\beta x_2)$	$\left(\pi^2-\frac{1}{4}\left(\alpha^2-\beta^2\right)\right)^{1/2}$	
(17) $\mu_0(1+\alpha x_2)\exp(\pm\beta x_1)$	$\left(\pi^2-\frac{1}{4}\left(\alpha^2-\beta^2\right)\right)^{1/2}$	
(18) $\mu_0(1+\alpha x_1)^n\exp(\pm\beta x_2)$	$\left[\pi^2-\frac{1}{4}\left(\alpha^2 n(2-n)-\beta^2\right)\right]^{1/2}$,	if $n\in(0,2)$
	$\left(\pi^2+\frac{1}{4}\beta^2\right)^{1/2}>\pi$,	if $n\notin(0,2)$
(19) $\mu_0(1+\alpha x_2)^n\exp(\pm\beta x_1)$	$\left[\pi^2-\frac{1}{4}\left(\alpha^2 n(2-n)-\beta^2\right)\right]^{1/2}$,	if $n\in(0,2)$
	$\left(\pi^2+\frac{1}{4}\beta^2\right)^{1/2}>\pi$,	if $n\in\{0,2\}$
	$\left[\pi^2+\frac{1}{4}\left(\alpha^2 n(n-2)(1+\alpha)^{-2}+\beta^2\right)\right]^{1/2}>\pi$,	if $n\notin[0,2]$

Table 3
Conditions on α, β and n

μ	Conditions on α,β,n in order (3.18) holds	Conditions on α,β,n in order (3.26) is implied by (3.2)	
(2)	$\alpha^2 \leqslant 4\pi^2$		
(4)	$\alpha^2 n(2-n) \leqslant 4\pi^2$, if $n \in (0,2)$		
(7)	$\alpha^2 \leqslant 4\pi^2$		
(9)	$\alpha^2 n(2-n) \leqslant 4\pi^2$, if $n \in (0,2)$		
(10)		$3\alpha^2 \leqslant 4\pi^2$	
(12)	$\alpha^2 + \beta^2 \leqslant 4\pi^2$		
(14)	$(\alpha^2+\beta^2)n(2-n) \leqslant 4\pi^2$, if $n \in (0,2)$		
(15)		$3\alpha^2 - \beta^2 \leqslant 4\pi^2$	
(16)	$\alpha^2 - \beta^2 \leqslant 4\pi^2$		
(17)	$\alpha^2 - \beta^2 \leqslant 4\pi^2$	$\alpha^2 + 3\beta^2 \leqslant 4\pi^2$	
(18)	$\alpha^2 n(2-n) - \beta^2 \leqslant 4\pi^2$, if $n \in (0,2)$		
(19)	$\alpha^2 n(2-n) - \beta^2 \leqslant 4\pi^2$, if $n \in (0,2)$	$n(2-n)\alpha^2 + 3\beta^2 \leqslant 4\pi^2$,	if $n \in (0,2)$
		$3\beta^2 \leqslant 4\pi^2$,	if $n \in \{0,2\}$
		$n(2-n)(1-\alpha)^{-2}\alpha^2 + 3\beta^2 \leqslant 4\pi^2$,	if $n \notin [0,2]$

From (3.19) the rate of decay depends on

$$\sup_{L(\zeta)} M(\Lambda^{-1/2}) \exp(-2k\zeta), \tag{4.8}$$

where k is given by (3.17). If the term $\Lambda^{-1/2}$ does not depend of x_1, then it does not contribute to the asymptotic behavior when $x_1 \to +\infty$ and so (3.19) is directly comparable with (3.16). See examples 1–5 of Table 1. If the term $\Lambda^{-1/2}$ depends on x_1, we have to take into account the contribution to the decay rate. See examples 6–19 of Tables 1 and 2. In the majority of the cases we consider the condition (3.26) is guaranteed whenever (3.2) holds. However, in some cases this fact does not happen. We impose the conditions (when it is necessary) in Table 3.

We have seen that the rate of decay in the case of mixtures of type (4.1) is very similar to the one of the equation $(\mu(x_1,x_2)\ u_{,i})_{,i} = 0$.

A natural question is to compare the rate of decay obtained for nonhomogeneous materials with the case of homogeneous materials. We know that in the homogeneous case the rate of decay is π. Thus, we can compare the rate of decay with the help of the Tables 1 and 2.

5. Discussion: case (3.7)

In this section, we apply the arguments of the previous section to the case that the matrix Λ is defined by (3.7), with $\mu_{11}(x_1,x_2) > 0, \mu_{22}(x_1,x_2) > 0$ for $x_1 \in [0,+\infty), x_2 \in [0,1]$. In this case, we have

$$\Lambda^{-1/2} = \begin{pmatrix} \mu_{11}^{-1/2} & 0 \\ 0 & \mu_{22}^{-1/2} \end{pmatrix}, \quad \Lambda(\Lambda^{-1/2})_{,i} = -\frac{1}{2}\begin{pmatrix} \mu_{11}^{-1/2}\mu_{11,i} & 0 \\ 0 & \mu_{22}^{-1/2}\mu_{22,i} \end{pmatrix}. \tag{5.1}$$

Thus

$$\Lambda^{-1/2}\left[\Lambda(\Lambda^{-1/2})_{,i}\right]_{,i} = \frac{1}{4}\begin{pmatrix} \mu_{11}^{-2}(\mu_{11,i}\mu_{11,i} - 2\mu_{11}\mu_{11,ii}) & 0 \\ 0 & \mu_{22}^{-2}(\mu_{22,i}\mu_{22,i} - 2\mu_{22}\mu_{22,ii}) \end{pmatrix}. \tag{5.2}$$

On the other hand

$$\Lambda^{-1/2} a \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix} \Lambda^{-1/2} = a \begin{pmatrix} -\mu_{11}^{-1} & \mu_{11}^{-1/2}\mu_{22}^{-1/2} \\ \mu_{11}^{-1/2}\mu_{22}^{-1/2} & -\mu_{22}^{-1} \end{pmatrix}. \tag{5.3}$$

We have that

$$\Gamma = \begin{pmatrix} -\mu_{11}^{-1/2}\Delta\mu_{11}^{1/2} - a\mu_{11}^{-1} & a\mu_{11}^{-1/2}\mu_{22}^{-1/2} \\ a\mu_{11}^{-1/2}\mu_{22}^{-1/2} & -\mu_{22}^{-1/2}\Delta\mu_{22}^{1/2} - a\mu_{22}^{-1} \end{pmatrix}. \tag{5.4}$$

We claim that

(i) If $\Delta\mu_{11}^{1/2} \geqslant 0$, $\Delta\mu_{22}^{1/2} \geqslant 0$ and $a \geqslant 0$, then $\Gamma^+ \leqslant 0$ and the rate of decay of the functions (U, W) is greater than or equal to that of the Laplace's equation (or the homogeneous case).
(ii) If $\Delta\mu_{11}^{1/2} \leqslant 0, \Delta\mu_{22}^{1/2} \leqslant 0$ and $a \geqslant 0$, then $\Gamma^+ \geqslant 0$. So, the rate of decay of the functions (U, W) is less than or equal to that for Laplace's equation (or the homogeneous case).
(iii) Otherwise, i.e. if $\Delta\mu_{11}^{1/2} \cdot \Delta\mu_{22}^{1/2} \leqslant 0$, and $a \geqslant 0$, the sign of Γ^+ depends on each case.

Now we prove the claim:

First we prove (i). Since $-\mu_{11}^{-1/2}\Delta\mu_{11}^{1/2} - a\mu_{11}^{-1} \leqslant 0$ and $\det(\Gamma) \geqslant 0$, Γ is semi-definite negative and so $\Gamma^+ \leqslant 0$.

Now we prove (ii). When the minor $-\mu_{11}^{-1/2}\Delta\mu_{11}^{1/2} - a\mu_{11}^{-1} \geqslant 0$, then $\Gamma^+ \geqslant 0$ and (ii) is satisfied. Therefore we can suppose that $-\mu_{11}^{-1/2}\Delta\mu_{11}^{1/2} - a\mu_{11}^{-1} \leqslant 0$. Thus we have $\mu_{11}^{1/2}\Delta\mu_{11}^{1/2} + a \geqslant 0$. It is sufficient to see that $\det(\Gamma) \leqslant 0$. From the hypothesis $\mu_{11}^{-1/2}\Delta\mu_{22}^{1/2} + \mu_{22}^{-1/2}\Delta\mu_{11}^{1/2} \leqslant 0$. Then

$$(\mu_{11}^{1/2}\Delta\mu_{11}^{1/2} + a)(\mu_{11}^{-1/2}\Delta\mu_{22}^{1/2} + \mu_{22}^{-1/2}\Delta\mu_{11}^{1/2}) \leqslant 0. \tag{5.5}$$

Thus $\Delta\mu_{11}^{1/2}\Delta\mu_{22}^{1/2} + \mu_{11}^{1/2}\mu_{22}^{-1/2}\left(\Delta\mu_{11}^{1/2}\right)^2 + a\left(\mu_{11}^{-1/2}\Delta\mu_{22}^{1/2} + \mu_{22}^{-1/2}\Delta\mu_{11}^{1/2}\right) \leqslant 0$. It follows:

$$\Delta\mu_{11}^{1/2}\Delta\mu_{22}^{1/2} + a(\mu_{11}^{-1/2}\Delta\mu_{22}^{1/2} + \mu_{22}^{-1/2}\Delta\mu_{11}^{1/2}) \leqslant -\mu_{11}^{1/2}\mu_{22}^{-1/2}(\Delta\mu_{11}^{1/2})^2 \leqslant 0.$$

We note that

$$\det(\Gamma) = \mu_{11}^{-1/2}\mu_{22}^{-1/2}\left[\Delta\mu_{11}^{1/2}\Delta\mu_{22}^{1/2} + a\left(\mu_{11}^{-1/2}\Delta\mu_{22}^{1/2} + \mu_{22}^{-1/2}\Delta\mu_{11}^{1/2}\right)\right] \leqslant 0. \tag{5.6}$$

Finally, to prove (iii) we present two examples such that $\Delta\mu_{11}^{1/2} \cdot \Delta\mu_{22}^{1/2} \leqslant 0$, and $a \geqslant 0$; one of them with $\Gamma^+ > 0$ and the other one with $\Gamma^+ < 0$.

Example 5.1. Let us consider $\mu_{11} = \mu_0(1 + x_2)^{-1}$, $\mu_{22} = \mu_0(1 + x_2)$ and $a = \frac{1}{4}\mu_0(1 + x_2)^{-1}$. Then $\Delta\mu_{11}^{1/2} = \frac{3}{4}\mu_0^{1/2}(1 + x_2)^{-5/2} > 0$ and $\Delta\mu_{22}^{1/2} = -\frac{1}{4}\mu_0^{1/2}(1 + x_2)^{-3/2} < 0$. In this case

$$\Gamma = \begin{pmatrix} -\frac{3}{4}(1 + x_2)^{-2} - \frac{1}{4} & \frac{1}{4}(1 + x_2)^{-1} \\ \frac{1}{4}(1 + x_2)^{-1} & 0 \end{pmatrix}. \tag{5.7}$$

From (3.3) the maximum eigenvalue of Γ is

$$M(\Gamma) = \frac{1}{2}\left(\frac{-3}{4}(1 + x_2)^{-2} - \frac{1}{4} + \sqrt{\left(\frac{-3}{4}(1 + x_2)^{-2} - \frac{1}{4}\right)^2 + \frac{1}{4}(1 + x_2)^{-2}}\right). \tag{5.8}$$

Since $(\frac{-3}{4}(1 + x_2)^{-2} - \frac{1}{4})^2 < (\frac{-3}{4}(1 + x_2)^{-2} - \frac{1}{4})^2 + \frac{1}{4}(1 + x_2)^{-2}$, it follows that $M(\Gamma) > 0$, for all $x_2 \in [0, 1]$ and so $\Gamma^+ > 0$. Furthermore, since $x_2 \in [0, 1]$, we have $M(\Gamma) \leqslant \frac{1}{4}(\sqrt{5} - \frac{7}{8})$ and so $\Gamma^+ < \pi^2$.

Example 5.2. Let us consider

$$\mu_{11}=\exp(x_2),\quad \mu_{22}=1+\frac{1}{2}x_2\quad\text{and}\quad a=\frac{\exp(x_2)}{4\left(1+\frac{1}{2}x_2\right)\exp(x_2)-1}. \tag{5.9}$$

Then $\Delta\mu_{11}^{1/2}=\frac{1}{4}\exp(x_2/2)>0$ and $\Delta\mu_{22}^{1/2}=-\frac{1}{16}\left(1+\frac{1}{2}x_2\right)^{-3/2}<0$. The matrix Γ takes the form

$$\Gamma=\begin{pmatrix} -\frac{1}{4}-\frac{1}{4\left(1+\frac{1}{2}x_2\right)\exp(x_2)-1} & \frac{\exp(x_2/2)\left(1+\frac{1}{2}x_2\right)^{-1/2}}{4\left(1+\frac{1}{2}x_2\right)\exp(x_2)-1} \\ \frac{\exp(x_2/2)\left(1+\frac{1}{2}x_2\right)^{-1/2}}{4\left(1+\frac{1}{2}x_2\right)\exp(x_2)-1} & \frac{1}{16}\left(1+\frac{x_2}{2}\right)^{-2}-\frac{\exp(x_2)\left(1+\frac{1}{2}x_2\right)^{-1}}{4\left(1+\frac{1}{2}x_2\right)\exp(x_2)-1} \end{pmatrix}. \tag{5.10}$$

The first minor of Γ is

$$-\frac{1}{4}-\frac{1}{4\left(1+\frac{x_2}{2}\right)\exp(x_2)-1}<0\quad\text{for all } x_2\in[0,1] \tag{5.11}$$

and $\det(\Gamma)$ is

$$\frac{1}{4\left(1+\frac{1}{2}x_2\right)\exp(x_2)-1}\left[\frac{1}{4}\exp(x_2)\left(1+\frac{x_2}{2}\right)^{-1}-\frac{1}{16}\left(1+\frac{x_2}{2}\right)^{-2}\right]-\frac{1}{4^3}\left(1+\frac{x_2}{2}\right)^{-2}>0 \tag{5.12}$$

for all $x_2\in[0,1]$. Therefore Γ is defined negative and so $M(\Gamma)<0$, for all $x_2\in[0,1]$. Thus $\Gamma^+<0$. So the claim is proved.

Despite we have seen the behavior of the functions (U,W), we know that this is not sufficient to clarify the behavior of the problem for the unknowns (u,w). As in the previous section, we need to pay attention to guarantee that when the solution (u,w) satisfies conditions (3.1) and (3.2) then (3.26) holds and the rate of decay will be determined by (4.8). In Examples 5.1 and 5.2 this is not a problem because the functions μ_{11} and μ_{22} only depend on the variable x_2. We think that the best way to clarify these aspect could be by means of some examples.

Example 5.3. Set $\mu_{11}=\mu(x_1,x_2)$ and $\mu_{22}=\lambda\mu(x_1,x_2)$, $\lambda>0$, then $M(\Gamma)=-\mu^{-1/2}\Delta\mu^{1/2}$. Of course, this is a particular case of (3.6) where $k_{11}=1, k_{22}=\lambda$ and $k_{12}=0$. The discussion concerning condition (3.26) can be done as in Section 6.

Example 5.4. Consider $\mu_{11}=\mu(x_1)=\mu$ and $\mu_{22}=(1+\lambda x_2)^2\,\mu,\lambda>0$, then $M(\Gamma)=-\mu^{-1/2}\Delta\mu^{1/2}$.

We have $\Delta\mu_{22}^{1/2}=(1+\lambda x_2)\Delta\mu^{1/2}$ and

$$\Gamma=\begin{pmatrix} -\mu^{-1/2}\Delta\mu^{1/2}-a\mu^{-1} & a(1+\lambda x_2)^{-1}\mu^{-1} \\ a(1+\lambda x_2)^{-1}\mu^{-1} & -\mu^{-1/2}\Delta\mu^{1/2}-a(1+\lambda x_2)^{-2}\mu^{-1} \end{pmatrix} \tag{5.13}$$

$$\begin{aligned} M(\Gamma)&=\frac{1}{2}\left(-2\mu^{-1/2}\Delta\mu^{1/2}-a\mu^{-1}-a(1+\lambda x_2)^{-2}\mu^{-1}+\sqrt{\left(-a\mu^{-1}+a(1+\lambda x_2)^{-2}\mu^{-1}\right)^2+4a^2(1+\lambda x_2)^{-2}\mu^{-2}}\right)\\ &=\frac{1}{2}\left(-2\mu^{-1/2}\Delta\mu^{1/2}-a\mu^{-1}-a(1+\lambda x_2)^{-2}\mu^{-1}+\sqrt{\left(a\mu^{-1}+a(1+\lambda x_2)^{-2}\mu^{-1}\right)^2}\right)\\ &=-\mu^{-1/2}\Delta\mu^{1/2}, \end{aligned} \tag{5.14}$$

independently of a and λ.

We note that by symmetry, if $\mu_{11} = \mu(x_2) = \mu$ and $\mu_{22} = (1 + \lambda x_1)^2\ \mu, \lambda > 0$, then $M(\Gamma) = -\mu^{-1/2}\Delta\mu^{1/2}$. See the examples of Tables 1 and 2, where we compute the rate of decay for some functions from $M(\Gamma)$.

Now we explain some examples of functions and we obtain some bounds for the $M(\Gamma)$. In the two following examples we get $M(\Gamma) \leqslant -\mu^{-1/2}\Delta\mu^{1/2}$. The discussion concerning when the condition (3.26) holds can be done as in the Section 6. It would depend on the particular form of the function $\mu(x_1)$.

Example 5.5. Consider $\mu_{11} = \mu(x_1) = \mu$ and $\mu_{22} = (1 + \lambda x_2)^n\mu, \lambda > 0, n \geqslant 2$, then $M(\Gamma) \leqslant -\mu^{-1/2}\Delta\mu^{1/2}$. If $n = 2$, then in the above example $M(\Gamma)$ is exactly $-\mu^{-1/2}\Delta\mu^{1/2}$.

We have

$$\mu_{22}^{1/2} = (1 + \lambda x_2)^{n/2}\mu^{1/2} \text{ and } \Delta\mu_{22}^{1/2} = (1 + \lambda x_2)^{n/2}\Delta\mu^{1/2} + \frac{n}{2}\left(\frac{n}{2} - 1\right)(1 + \lambda x_2)^{(n/2)-2}\lambda^2\mu^{1/2}.$$

So

$$\Gamma = \begin{pmatrix} -\mu^{-1/2}\Delta\mu^{1/2} - a\mu^{-1} & a(1 + \lambda x_2)^{-n/2}\mu^{-1} \\ a(1 + \lambda x_2)^{-n/2}\mu^{-1} & -\mu^{-1/2}\Delta\mu^{1/2} - \frac{n}{2}\left(\frac{n}{2} - 1\right)(1 + \lambda x_2)^{-2}\lambda^2 - a(1 + \lambda x_2)^{-n}\mu^{-1} \end{pmatrix}. \tag{5.15}$$

Thus

$$M(\Gamma) = \frac{1}{2}\left[-2\mu^{-1/2}\Delta\mu^{1/2} - a\mu^{-1} - \frac{n(n-2)}{4}(1 + \lambda x_2)^{-2}\lambda^2 - a(1 + \lambda x_2)^{-n}\mu^{-1} \right.$$
$$\left. + \left(\left(-a\mu^{-1} + \frac{n(n-2)}{4}(1 + \lambda x_2)^{-2}\lambda^2 + a(1 + \lambda x_2)^{-n}\mu^{-1}\right)^2 + 4a^2(1 + \lambda x_2)^{-n}\mu^{-2}\right)^{1/2}\right] \tag{5.16}$$

and

$$M(\Gamma) = \frac{1}{2}\left[-2\mu^{-1/2}\Delta\mu^{1/2} - a\mu^{-1} - \frac{n(n-2)}{4}(1 + \lambda x_2)^{-2}\lambda^2 - a(1 + \lambda x_2)^{-n}\mu^{-1} \right.$$
$$+ \left((a\mu^{-1} + a(1 + \lambda x_2)^{-n}\mu^{-1})^2 + \left(\frac{n(n-2)}{2}\right)^2(1 + \lambda x_2)^{-4}\lambda^4\right.$$
$$\left.\left. + a\frac{n(n-2)}{2}\lambda^2(1 + \lambda x_2)^{-2}\mu^{-1}((1 + \lambda x_2)^{-n} - 1)\right)^{1/2}\right] \tag{5.17}$$

Since $n \geqslant 2,\ a\frac{n(n-2)}{2}\lambda^2(1 + \lambda x_2)^{-2}\mu^{-1}((1 + \lambda x_2)^{-n} - 1) \leqslant 0$ and so

$$M(\Gamma) \leqslant \frac{1}{2}\left[-2\mu^{-1/2}\Delta\mu^{1/2} - a\mu^{-1} - \frac{n(n-2)}{4}(1 + \lambda x_2)^{-2}\lambda^2 - a(1 + \lambda x_2)^{-n}\mu^{-1} \right.$$
$$\left. + \left((a\mu^{-1} + a(1 + \lambda x_2)^{-n}\mu^{-1})^2 + \left(\frac{n(n-2)}{2}\right)^2(1 + \lambda x_2)^{-4}\lambda^4\right)^{1/2}\right]$$
$$\leqslant -\mu^{-1/2}\Delta\mu^{1/2}. \tag{5.18}$$

We note that by symmetry, if $\mu_{11} = \mu(x_2) = \mu$ and $\mu_{22} = (1 + \lambda x_1)^n\mu, \lambda > 0,\ n \geqslant 2$, then $M\ (\Gamma) \leqslant -\ \mu^{-1/2}\Delta\mu^{1/2}$. Thus the rate of decay k will be greater or equal to the one proposed in the tables. Again, to guarantee that condition (3.26) holds depend on the particular form of the function μ. In the Table 3, we can see sufficient conditions to guarantee the asymptotic behavior.

Example 5.6. Consider $\mu_{11} = \mu(x_1) = \mu$ and $\mu_{22} = \rho^2\exp(2\lambda x_2)\mu, \lambda > 0, \rho \geqslant 1$, then $M(\Gamma) \leqslant -\mu^{-1/2}\Delta\mu^{1/2}$.

We have $\mu_{22}^{1/2} = \rho\exp(\lambda x_2)\mu^{1/2}$ and $\Delta\mu_{22}^{1/2} = \rho\exp(\lambda x_2)\left(\lambda^2\mu^{1/2} + \Delta\mu^{1/2}\right)$.

$$\Gamma = \begin{pmatrix} -\mu^{-1/2}\Delta\mu^{1/2} - a\mu^{-1} & a\rho^{-1}\exp(-\lambda x_2)\mu^{-1} \\ a\rho^{-1}\exp(-\lambda x_2)\mu^{-1} & -\mu^{-1/2}\Delta\mu^{1/2} - \lambda^2 - a\rho^{-2}\exp(-2\lambda x_2)\mu^{-1} \end{pmatrix}. \tag{5.19}$$

Then

$$\begin{aligned} M(\Gamma) &= \frac{1}{2}\Bigg(-2\mu^{-1/2}\Delta\mu^{1/2} - a\mu^{-1} - \lambda^2 - a\rho^{-2}\exp(-2\lambda x_2)\mu^{-1} \\ &\quad + \sqrt{\left(-a\mu^{-1} + \lambda^2 + a\rho^{-2}\exp(-2\lambda x_2)\mu^{-1}\right)^2 + 4a^2\mu^{-2}\rho^{-2}\exp(-2\lambda x_2)}\Bigg) \\ &= \frac{1}{2}\Bigg(-2\mu^{-1/2}\Delta\mu^{1/2} - a\mu^{-1} - \lambda^2 - a\rho^{-2}\exp(-2\lambda x_2)\mu^{-1} \\ &\quad + \sqrt{(-a\mu^{-1} + a\rho^{-2}\exp(-2\lambda x_2)\mu^{-1})^2 + \lambda^4 + 2a\lambda^2\mu^{-1}(\rho^{-2}\exp(-2\lambda x_2) - 1)}\Bigg) \end{aligned} \tag{5.20}$$

Since $\rho \geqslant 1, 2a\lambda^2\mu^{-1}(\rho^{-2}\exp(-2\lambda\ x_2) - 1) \leqslant 0$ and so

$$\begin{aligned} M(\Gamma) &\leqslant \frac{1}{2}\left(-2\mu^{-1/2}\Delta\mu^{1/2} - a\mu^{-1} - \lambda^2 - a\rho^{-2}\exp(-2\lambda x_2)\mu^{-1} + \sqrt{(-a\mu^{-1} + a\rho^{-2}\exp(-2\lambda x_2)\mu^{-1})^2 + \lambda^4}\right) \\ &\leqslant -\mu^{-1/2}\Delta\mu^{1/2}. \end{aligned} \tag{5.21}$$

We note that by symmetry, if $\mu_{11} = \mu(x_2) = \mu$ and $\mu_{22} = \rho^2\exp(2\lambda x_1)\mu$, $\lambda > 0$, $\rho \geqslant 1$, then $M(\Gamma) \leqslant -\mu^{-1/2}\Delta\mu^{1/2}$. A similar comment to the one at the end of Example 5.5 can be made in this case. It is worth noting that when $\mu_{22} = \rho^2\exp(2\lambda x_1)\ \mu, \lambda > 0$, a further analysis (on λ) is necessary to guarantee that condition (3.26) holds. We left it as an easy exercise.

Example 5.7. Consider $\mu_{11} = \lambda$ constant, $\lambda > 0$ and $\mu_{22} = \mu(x_1, x_2)$. The matrix Γ is

$$\Gamma = \begin{pmatrix} -a\lambda^{-1} & a\lambda^{-1/2}\mu^{-1/2} \\ a\lambda^{-1/2}\mu^{-1/2} & -\mu^{-1/2}\Delta\mu^{1/2} - a\mu^{-1} \end{pmatrix},$$

$$\begin{aligned} M(\Gamma) &= \frac{1}{2}\left(-a(\lambda^{-1} + \mu^{-1}) - \mu^{-1/2}\Delta\mu^{1/2} + \sqrt{[a(\mu^{-1} - \lambda^{-1}) + \mu^{1/2}\Delta\mu^{1/2}]^2 + 4a^2\lambda^{-1}\mu^{-1}}\right) \\ &= \frac{1}{2}\left(-a\left(\lambda^{-1} + \mu^{-1}\right) - \mu^{-1/2}\Delta\mu^{1/2} + \sqrt{a^2(\mu^{-1} + \lambda^{-1})^2 + (\mu^{1/2}\Delta\mu^{1/2})^2 + 2a(\mu^{-1} - \lambda^{-1})\mu^{1/2}\Delta\mu^{1/2}}\right). \end{aligned}$$

If $\Delta\mu^{1/2} \geqslant 0$, then $M(\Gamma) \leqslant 0$. In fact

$$\begin{aligned} M(\Gamma) &\leqslant \frac{1}{2}\left(-a\left(\lambda^{-1} + \mu^{-1}\right) - \mu^{-1/2}\Delta\mu^{1/2} + \sqrt{a^2(\mu^{-1} + \lambda^{-1})^2 + (\mu^{1/2}\Delta\mu^{1/2})^2 + 2a(\mu^{-1} + \lambda^{-1})\mu^{1/2}\Delta\mu^{1/2}}\right) \\ &= 0 \end{aligned}$$

This result agrees with the above claim in the current section.

If $\Delta\mu^{1/2} \leqslant 0$, then $M(\Gamma) \leqslant -\mu^{-1/2}\Delta\mu^{1/2}$. In fact

$$M(\Gamma) = \frac{1}{2}\left(-a(\lambda^{-1}+\mu^{-1}) - \mu^{-1/2}\Delta\mu^{1/2} + \sqrt{a^2(\mu^{-1}+\lambda^{-1})^2 + (\mu^{1/2}\Delta\mu^{1/2})^2 - 2a(\lambda^{-1}-\mu^{-1})\mu^{1/2}\Delta\mu^{1/2}}\right)$$
$$\leqslant \frac{1}{2}\left(-a(\lambda^{-1}+\mu^{-1}) - \mu^{-1/2}\Delta\mu^{1/2} + \sqrt{\left[a(\mu^{-1}+\lambda^{-1}) - \mu^{-1/2}\Delta\mu^{1/2}\right]^2}\right) = -\mu^{-1/2}\Delta\mu^{1/2}.$$

Again the discussion whether condition (3.26) holds implies an analysis on the function μ. But this can be find in the Table 3.

6. Decay to the homogeneous displacement in the case (3.6)

In this section, we study the rate of decay to the homogeneous distribution of the solutions of the system (2.33) and (2.34) in the case of

$$\mu_{ij}(x_1,x_2) = k_{ij}\mu(x_1,x_2), \quad a = a(x_1,x_2), \tag{6.1}$$

where k_{ij} are constants such that the matrix

$$\begin{pmatrix} k_{11} & k_{12} \\ k_{12} & k_{22} \end{pmatrix} \tag{6.2}$$

is definite positive.

The main aim of this section is the study of the asymptotic behavior of the function $z = u - w$. First, we note that z satisfies the partial differential equation

$$(\mu(x_1,x_2)z_{,i})_{,i} - \delta(x_1,x_2)z = 0, \tag{6.3}$$

where

$$\delta(x_1,x_2) = \frac{k_{11}+k_{22}+2k_{12}}{k_{11}k_{22}-k_{12}^2}a(x_1,x_2) \tag{6.4}$$

and the boundary and asymptotic conditions

$$z = 0, \quad (x_1,x_2) \in [0,\infty)\times\{0,1\}, \quad z, z_{,i} \to 0 \quad \text{uniformly as } x_1 \to \infty. \tag{6.5}$$

The arguments to study this problem are similar to the ones proposed in Section 4. They are an extension of the arguments proposed by Chan and Horgan (1998). It is worth noting that this equation is different from the one studied by Chan and Horgan. This introduces new difficulties to study. We sketch the first step and we describe in a wider way the new arguments that this new question poses. We define the function $Z(x_1,x_2)$ by

$$z(x_1,x_2) = Z(x_1,x_2)\mu^{-1/2}(x_1,x_2). \tag{6.6}$$

We have that Z satisfies the equation

$$\Delta Z + \Upsilon(x_1,x_2)Z = 0, \tag{6.7}$$

where

$$\Upsilon(x_1,x_2) = \frac{1}{4}\mu^{-2}(\mu_{,i}\mu_{,i} - 2\mu\mu_{,ii}) - \delta\mu^{-1} = -\mu^{-1/2}[\Delta\mu^{1/2} + \delta\mu^{-1/2}]. \tag{6.8}$$

The asymptotic behavior of the solutions of the problem (6.3)–(6.5) may now be deduced from the asymptotic behavior of solutions Z of the problem (6.7) and (6.5). The exponential decay of solutions of (6.7) may be examined using energy-decay inequality methods. Consider first the case > 0. We can show that Z satisfies an inequality of the form

$$\int_{L(x_1)} Z^2 \mathrm{d}x_2 \leqslant K \exp(-2hx_1), \tag{6.9}$$

where

$$h^2 = \pi^2\left(1 - \frac{\Upsilon^+}{\pi^2}\right), \tag{6.10}$$

where the constant $^+$ is the maximum of the function (x_1, x_2) in the semi-infinite strip which is assumed strictly less than π^2. Thus from the change of variable (6.6), we deduce that z satisfies an inequality of the form

$$\int_{L(\zeta)} z^2 \leqslant K\mu^{-1}(x_1, x_2) \exp(-2hx_1), \tag{6.11}$$

where h is given by (6.10). When $\leqslant 0$, we can obtain the same estimate. It is worth noting that the nature of the right-hand side of (6.11) depends on the product

$$\mu^{-1}(x_1, x_2) \exp(-2hx_1). \tag{6.12}$$

It is worth giving some examples of functions μ to apply the previous results.

In order to guarantee that (6.9) holds when we assume (6.5), we need to guarantee that the condition

$$\lim_{z\to\infty} \exp(-2hz) \int_{L(z)} ZZ_{,1} \,\mathrm{d}x_2 = 0, \tag{6.13}$$

holds. This is the natural counterpart of the condition (3.26) in this situation. In the study of several examples we can see when this condition holds, but we do not discuss this aspect in this section.

Case 1. Let us to assume that $\mu = \mu(x_2), a = a(x_2)$. In this case

$$\Upsilon(x_2) = -\mu^{-1/2}\left[(\mu^{1/2})'' + \delta\mu^{-1/2}\right], \tag{6.14}$$

where the prime denotes $\mathrm{d}/\mathrm{d}x_2$ and so

$$\Upsilon^+ = \max_{[0,1]} \left[-\mu^{-1/2}\left((\mu^{1/2})'' + \delta\mu^{-1/2}\right)\right]. \tag{6.15}$$

Since μ and δ does not depend on x_1, the rate of decay is determined by $\exp(-2hx_1)$ of (6.11). Let us assume that $a(x_2) = \sigma\mu(x_2)$, where σ is a positive constant. Then the function becomes

$$\Upsilon(x_2) = -\mu^{-1/2}\left[(\mu^{1/2})'' + \overline{\delta}\mu^{1/2}\right] = -\mu^{-1/2}(\mu^{1/2})'' - \overline{\delta} \tag{6.16}$$

where $\overline{\delta}$ is the new constant

$$\overline{\delta} = \sigma \frac{k_{11} + k_{22} + 2k_{12}}{k_{11}k_{22} - k_{12}^2} > 0, \tag{6.17}$$

and satisfies $\overline{\delta} = \delta\mu^{-1}$. Thus

$$\Upsilon^+ = \max_{[0,1]} \left[-\mu^{-1/2}(\mu^{1/2})''\right] - \overline{\delta}. \tag{6.18}$$

We have

(i) If $(\mu^{1/2})'' \geqslant 0$, then $^{+} \leqslant 0$ and so the rate of decay given in (6.10) satisfies $h \geqslant \pi$. For example, we consider $\mu(x_2) = \mu_0 \exp(-\alpha x_2), \alpha > 0$. Here $(\mu^{1/2})'' = \frac{1}{4}\mu_0\alpha^2 \exp(-\frac{\alpha}{2}x_2)$ and $\Upsilon^{+} = -\frac{\alpha^2}{4} - \overline{\delta}$. Then, the decay-rate is $h = \left(\pi^2 + \frac{\alpha^2}{4} + \overline{\delta}\right)^{1/2} > \pi$.

(ii) If $(\mu^{1/2})'' = 0$, then $\mu(x_2) = \mu_0(1 + \alpha x_2)^2$. Moreover $\Upsilon^{+} = -\overline{\delta} < 0$. Therefore

$$h = (\pi^2 + \overline{\delta})^{1/2} = \left(\pi^2 + \frac{k_{11} + k_{22} + 2k_{12}}{k_{11}k_{22} - k_{12}^2}\sigma\right)^{1/2} > \pi. \tag{6.19}$$

Furthermore,

$$z(x_1, x_2) = (1 + \alpha x_2)^{-1} \exp(-hx_1) \sin(\pi x_2) \quad \text{with} \quad h = (\pi^2 + \overline{\delta})^{1/2} \tag{6.20}$$

is a solution of (6.3),(6.5).

(iii) If $(\mu^{1/2})'' \leqslant 0$, then the sign of $^{+}$ depends on each case.

Set $\mu(x_2) = 1 + \alpha x_2$ and $a(x_2) = \sigma(1 + \alpha x_2)$. Then $(\mu^{1/2})'' = -\frac{1}{4}\alpha^2(1 + \alpha x_2)^{-3/2}$ and $\Upsilon^{+} = \frac{1}{4}\alpha^2 - \overline{\delta}$. Now we shall consider two examples.

Example 6.1. For $\alpha = (\overline{\delta})^{1/2} = \sigma^{1/2}\left(\frac{k_{11}+k_{22}+2k_{12}}{k_{11}k_{22}-k_{12}^2}\right)^{1/2}$ we have

$$\Upsilon^{+} = \frac{\overline{\delta}}{4} - \overline{\delta} = -\frac{3\overline{\delta}}{4} < 0. \tag{6.21}$$

Example 6.2. For $\alpha = \left(4\overline{\delta} + 2\pi\right)^{1/2}$ we obtain

$$\Upsilon^{+} = \frac{1}{4}(4\overline{\delta} + 2\pi) - \overline{\delta} = \frac{\pi}{2} > 0. \tag{6.22}$$

Finally we shall consider the special case when $\mu = \mu(x_2)$, $\delta = \overline{\delta}\mu$ and

$$(\mu^{1/2})'' + \psi\mu^{1/2} = 0 \tag{6.23}$$

for some constants $\overline{\delta} > 0$ and ψ a real number. Since $\delta = \overline{\delta}\mu$, it follows that:

$$\Upsilon = -\mu^{-1/2}[(\mu^{1/2})'' + \delta\mu^{-1/2}] = -\mu^{-1/2}(\mu^{1/2})'' - \overline{\delta}. \tag{6.24}$$

From (6.23), $\Upsilon = \psi - \overline{\delta}$ is a constant, and so

$$\Upsilon^{+} = \psi - \overline{\delta}. \tag{6.25}$$

The solutions of the linear equation (6.22) are

(i) If $\psi = 0$, $\mu^{1/2} = \mu_0^{1/2}(1 + \alpha x_2)$ and $h = (\pi^2 + \overline{\delta})^{1/2}$.

(ii) If $\psi \neq 0$, $\mu^{1/2} = \mu_0 \exp(\pm\sqrt{-\psi}x_2)$ and $h = (\pi^2 - \psi + \overline{\delta})^{1/2}$.

In these examples the inhomogeneity is lateral. That means that only depends on x_2. Thus condition (6.13) is automatically satisfied when (6.5) holds.

Case 2. Let us to assume that $\mu = \mu(x_1)$, $a = a(x_1)$. In this case the term $\mu^{-1}(x_1)$ of (6.11) contributes to the asymptotic behavior when $x_1 \to +\infty$. Let us consider $a(x_1) = \sigma\mu(x_1)$. Then

$$\Upsilon = -\mu^{-1/2}(\mu^{1/2})'' - \bar{\delta} \tag{6.26}$$

where $\bar{\delta}$ is the constant

$$\bar{\delta} = \sigma \frac{k_{11} + k_{22} + 2k_{12}}{k_{11}k_{22} - k_{12}^2}. \tag{6.27}$$

We have

$$\Upsilon^+ = \sup_{[0,+\infty)} \left[-\mu^{-1/2}(\mu^{1/2})''\right] - \bar{\delta}. \tag{6.28}$$

(i) If $(\mu^{1/2})'' \geqslant 0$, then $^+ \leqslant 0$ and the decay rate h of (6.10) satisfies $h > \pi$. Consider, for instance, $\mu = \mu_0(1 + \alpha x_1)^{-1}$. Therefore $(\mu^{1/2})'' = \frac{3}{4}\mu_0^{1/2}\alpha^2(1 + \alpha x_1)^{-5/2} > 0$, and

$$\Upsilon^+ = \sup_{[0,+\infty)} \left[-\frac{3}{4}\alpha^2(1 + \alpha x_1)^{-2} - \bar{\delta}\right] = -\bar{\delta} < 0. \tag{6.29}$$

(ii) If $(\mu^{1/2})'' = 0$, then $\mu(x_1) = \mu_0(1 + \alpha x_1)^2$. Therefore $\Upsilon^+ = -\bar{\delta}$ and $h = (\pi^2 + \bar{\delta})^{1/2}$. Furthermore

$$z(x_1, x_2) = (1 + \alpha x_1)^{-1} \exp(-hx_1) \sin(\pi x_2), \quad \text{with } h = (\pi^2 + \bar{\delta})^{1/2} \tag{6.30}$$

is a solution of (6.3) and (6.5).

(iii) If $(\mu^{1/2})'' \leqslant 0$, then the sign of $^+$ depends of each case.

As the inhomogeneity with respect the variable x_1 is slower than any exponential, then condition (6.13) holds whenever (6.5) is satisfied.

Case 3. Let us to assume that $\mu = \mu(x_1, x_2)$. In this case the term $\mu^{-1}(x_1, x_2)\exp(-2hx_1)$ of (6.11) contributes to the asymptotic when $x_1 \to +\infty$. We study some examples.

Example 6.3. We consider $a(x_1, x_2) = \sigma\mu(x_1, x_2)$. Then the function Γ becomes

$$\Upsilon(x_1, x_2) = -\mu^{-1/2}\Delta\mu^{1/2} - \bar{\delta}, \quad \text{where } \bar{\delta} = \sigma\frac{k_{11} + k_{22} + 2k_{12}}{k_{11}k_{22} - k_{12}^2} > 0. \tag{6.31}$$

We have

$$\Upsilon^+ = \sup_{[0,+\infty)\times[0,1]} [-\mu^{-1/2}\Delta\mu^{1/2}] - \bar{\delta}. \tag{6.32}$$

(i) If $\Delta\mu^{1/2} \geqslant 0$, then $^+ \leqslant 0$ and the rate of decay of (6.10) satisfies $h > \pi$. Consider, for instance, $\mu(x_1, x_2) = \mu_0 \exp 2(\alpha x_1 + \beta x_2)$. Then $\Delta\mu^{1/2} = (\alpha^2 + \beta^2)\mu^{1/2} > 0$. So $\Upsilon = -\alpha^2 - \beta^2 - \bar{\delta} < 0$ is a constant. We obtain

$$h = (\pi^2 + \alpha^2 + \beta^2 + \bar{\delta})^{1/2} > \pi. \tag{6.33}$$

(ii) If $\Delta\mu^{1/2} = 0$, then $\Upsilon^{+} = -\overline{\delta}$. Therefore $h = (\pi^2 + \overline{\delta})^{1/2}$. We shall consider the example: $\mu(x_1, x_2) = \mu_0(1 + \alpha x_1 + \beta x_2)^2, \alpha > 0, \beta > 0$. In fact $\mu^{1/2}$ is harmonic. Furthermore

$$z(x_1, x_2) = (1 + \alpha x_1 + \beta x_2)^{-1} \exp(-hx_1) \sin(\pi x_2) \quad \text{with } h = (\pi^2 + \overline{\delta})^{1/2} \tag{6.34}$$

is a solution of (6.3) and (6.5).

(iii) If $(\Delta\mu^{1/2}) \leqslant 0$, then the sign of $^{+}$ depends of each case.

Example 6.4. Now we shall consider a particular example with $^{+} = 0$ and consequently $h = \pi$. We think over the general case when $\delta = \frac{k_{11}+k_{22}+2k_{12}}{k_{11}k_{22}-k_{12}^2}\, a(x_1, x_2)$.

Set $\mu(x_1, x_2) = \mu_0(1 + \alpha x_1 + \beta x_2)$, and $a(x_1, x_2) = \frac{1}{4}\, \mu_0\, \frac{k_{11}k_{22}-k_{12}^2}{k_{11}+k_{22}+2k_{12}}\, (\alpha^2 + \beta^2)(1 + \alpha x_1 + \beta x_2)^{-1}$. We compute that $(x_1, x_2) = 0$, and so $^{+} = 0$.

Remark. An alternative approach to the case 1 considered in this section is by means of the spectral arguments. When $\mu = \mu(x_2)$ and $a = \sigma\mu(x_2)$ the variable $z = u - v$ satisfies the equation

$$(\mu(x_2)z_{,i})_{,i} - \delta(x_2)z = 0, \quad z = 0, \ (x_1, x_2) \in [0, \infty) \times \{0, 1\}, \tag{6.35}$$

where

$$\delta = \sigma \frac{k_{11} + k_{22} + 2k_{12}}{k_{11}k_{22} - k_{12}^2} \tag{6.36}$$

The solutions will be combinations of the functions of the form

$$z = \exp(-\gamma x_1)g(x_2) \tag{6.37}$$

where $\gamma > 0$ in order to satisfy that the solutions tend to zero when x_1 tends to infinity and $g(x_2)$ is a solution of the eigenvalue

$$[\mu(x_2)g'(x_2)]' + \lambda\mu(x_2)g(x_2) = 0, \quad g(0) = g(1) = 0. \tag{6.38}$$

We can obtain that

$$\gamma = \left(\lambda + \sigma \frac{k_{11} + k_{22} + 2k_{12}}{k_{11}k_{22} - k_{12}^2}\right)^{1/2}, \tag{6.39}$$

where λ are the eigenvalues of the problem (6.38).

7. Conclusions

In this paper we have analyzed the spatial decay of anti-plane shear deformations in a mixture of isotropic but nonhomogeneous (in general) elastic solids. We have proposed a change of variable which has allowed to study some families of boundary value problems proposed by the anti-plane deformations. Basically, in this paper we have studied the case that corresponds to mixtures defined by examples (3.6) and (3.7).

In the case problem determined by matrices of the type (3.6) we have proved that the rate of decay is similar to the case of elastic materials where the inhomogeneity is defined by the function $\mu(x_1, x_2)$. Then the rate of decay k is for several examples of functions μ is given in the Tables 1 and 2. It is worth noting that in the case of the homogeneous isotropic mixture the rate of decay agrees with that of the Laplace equation. In the same tables, we can see the comparison with respect to it.

We also have studied the problem determined by matrices of the type (3.7). In this situation we have basically two inhomogeneities, one for each phase. First we have proved that when both inhomogeneities have a positive (or negative) Laplacian the behavior is similar to the one of the usual elasticity. But when the product of the Laplacians is negative, then there is no determined behavior and further analysis is

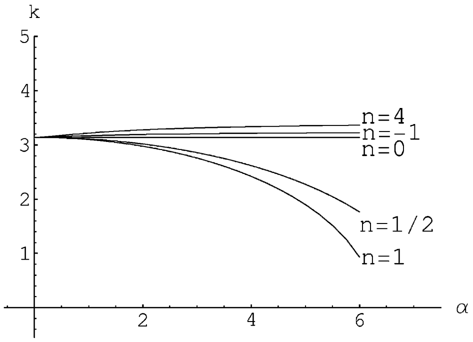

Fig. 2. Dependence of the rate of decay with respect to the parameters for $\mu = \mu_0(1 + \alpha x_2)^n$.

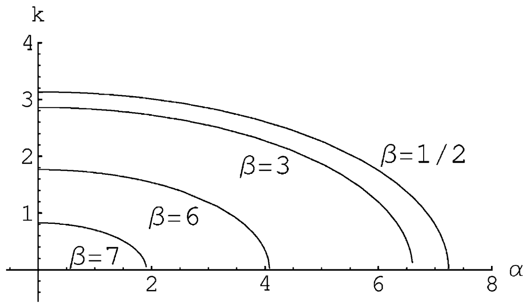

Fig. 3. Dependence of the rate of decay with respect to the parameters for $\mu = \mu_0(1 + \alpha x_1 + \beta x_2)^n$ with $n = \frac{1}{2}$.

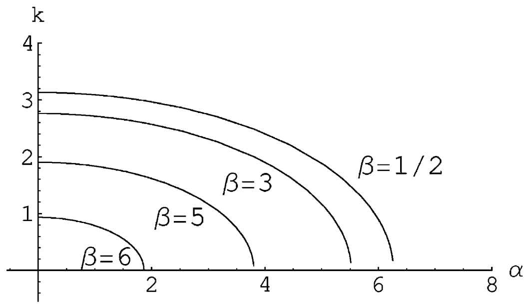

Fig. 4. Dependence of the rate of decay with respect to the parameters for $\mu = \mu_0(1 + \alpha x_1 + \beta x_2)^n$ with $n = 1$.

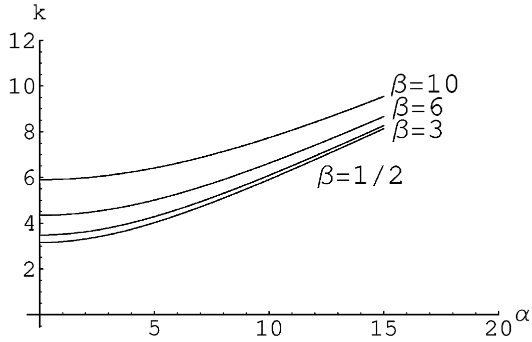

Fig. 5. Dependence of the rate of decay with respect to the parameters for $\mu = \mu_0 \exp(\pm \alpha x_1 \pm \beta x_2)$.

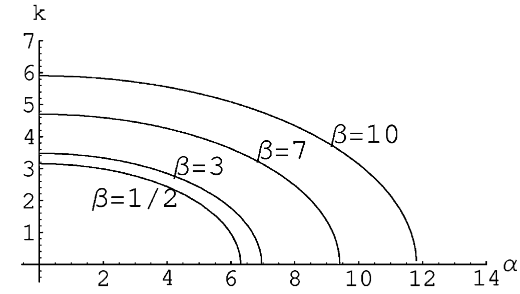

Fig. 6. Dependence of the rate of decay with respect to the parameters for $\mu = \mu_0(1 + \alpha x_1)\exp(\pm \beta x_2)$.

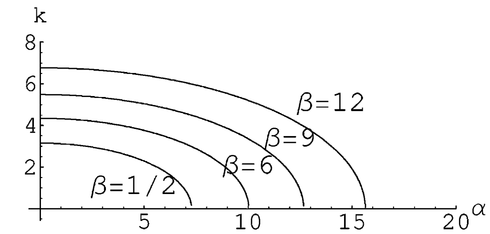

Fig. 7. Dependence of the rate of decay with respect to the parameters for $\mu = \mu_0(1 + \alpha x_2)^n \exp(\pm\beta x_1)$ with $n = \frac{1}{2}$.

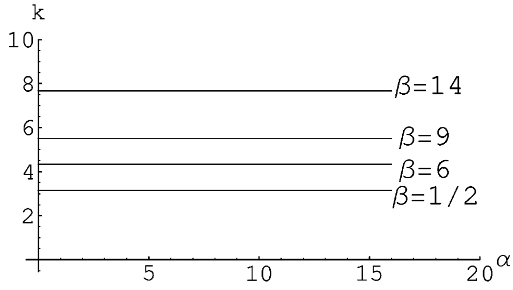

Fig. 8. Dependence of the rate of decay with respect to the parameters for $\mu = \mu_0(1 + \alpha x_2)^n \exp(\pm \beta x_1)$ with $n \in \{0,2\}$.

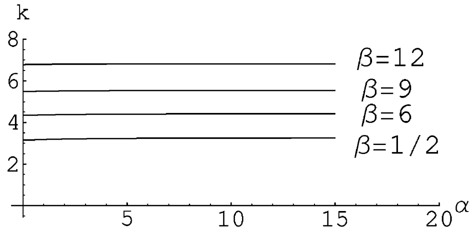

Fig. 9. Dependence of the rate of decay with respect to the parameters for $\mu = \mu_0(1 + \alpha x_2)^n \exp(\pm \beta x_1)$ with $n = 3$.

necessary. We also have considered some families of examples and we have given some estimates of the asymptotic behavior.

Section 6 is devoted to the study of the decay to the homogeneous displacement for the boundary value problem proposed when the both phases have the same kind of inhomogeneity. This problem proposes the study of a certain partial differential equation (see Eq. (6.3)). We have distinguished three kind of examples which depend on the inhomogeneity of the functions and we have obtained a bound for the rate of decay. To clarify the dependence of the rate of decay with respect of several parameters for some functions $\mu(x)$, we include eight pictures at the end of this paper (Figs. 2–9).

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